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DIRICHLET SERIES WHOSE PARTIAL SUMS OF COEFFICIENTS HAVE REGULAR VARIATION

BY

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ABSTRACT

We study pairs of Dirichlet series $A(s) = \sum_{n=1}^{\infty} a(n)n^{-s}$ and $P(s) = \sum_{n=2}^{\infty} p(n)n^{-s}$ in which $a(n)$ counts the number of objects of "size" n of some class of objects which is closed under formation of direct products and extraction of irreducible factors, and $p(n)$ counts the number of objects of "size" n which are irreducible in this class. We prove Dirichlet series analogues of certain results about power series and use these results to prove some conjectures of Burris concerning first-order 0–1 laws.

1. Introduction

Combinatorialists often study families of (unlabeled) objects which are closed under the process of formation of disjoint unions and extraction of connected components. For instance, the collection of all unlabeled finite graphs; the collection of unlabeled binary forests; and the collection of partitions of a number (here we think of a partition of the natural number n as "connected" if it has precisely one part) are all examples where this phenomenon occurs.

In such cases, if $c(n)$ denotes the number of connected objects of "size" n and $a(n)$ denotes the total number of objects of "size" n, then we have the following relation between an ordinary generating functions and an infinite product:

$$\sum_{n\geq 0} a(n)x^n = \prod_{j\geq 0}(1-x^j)^{-c(j)}. \tag{1.1}$$

There is a multiplicative analogue of this result, which can be described as follows. If one has a family of objects which is closed under the process of

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taking irreducible factors and taking finite products, a relation can be obtained between a Dirichlet series and an Euler product. We let $p(n)$ denote the number of irreducible objects of "size" n and let $a(n)$ denote the total number of objects of size n. Then we have

$$\sum_{n\geq 1} a(n)n^{-s} = \prod_{j\geq 2}(1-j^{-s})^{-p(j)}. \tag{1.2}$$

An example of where this phenomenon occurs is given by the collection of abelian groups (up to isomorphism). Every abelian group can be uniquely factored into a product of cyclic groups of prime power order. Thus $p(n)=1$ if n is a power of a prime and $p(n)=0$ otherwise. Hence if $a(n)$ is the number of abelian groups of order n, then

$$\sum_{n\geq 1} a(n)n^{-s} = \prod_{j\geq 2}(1-j^{-s})^{-p(j)} = \prod_{k\geq 1}\zeta(ks).$$

In [2], the following results for power series with nonnegative coefficients

$$A(x) := \sum_{n=0}^{\infty} a(n)x^n \quad \text{and} \quad C(x) := \sum_{n=0}^{\infty} c(n)x^n$$

satisfying

$$A(x) = \prod_{j=1}^{\infty}(1-x^j)^{-c(j)} \tag{1.3}$$

are given:

$$\text{if } c(n-1)/c(n) \to 1, \text{ then } a(n-1)/a(n) \to 1; \quad \text{and} \tag{1.4}$$

if $c(n-1)/c(n)\to\rho$ with $0<\rho<1$ and $\liminf_{n\to\infty} nc(n)r^n > 1$, then

$$a(n-1)/a(n)\to\rho \text{ and } a(n-1)/a(n) < \rho \text{ for all } n \text{ sufficiently large.} \tag{1.5}$$

These results were then applied to deduce that certain classes of structures have second-order logical limit laws. Here, we prove multiplicative analogues of these results, substituting Dirichlet series for power series. Unfortunately, there does not seem to be any easy way to lift the original proofs of the results for power series to the Dirichlet series case and so new techniques are developed.

Suppose

$$\mathbf{P}(s) = \sum_{n=2}^{\infty} p(n)n^{-s} \quad \text{and} \quad \mathbf{A}(s) = \sum_{n=1}^{\infty} a(n)n^{-s}$$

are Dirichlet series with nonnegative coefficients which satisfy

$$\mathbf{A}(s) = \prod_{j=2}^{\infty} (1 - j^{-s})^{-p(j)}. \tag{1.6}$$

We let

$$P(x) := \sum_{n \leq x} p(n) \quad \text{and} \quad A(x) = \sum_{n \leq x} a(n).$$

We prove the following results.

THEOREM 1: *Let $\mathbf{P}(s)$, $\mathbf{A}(s)$, $P(x)$, $A(x)$ be as above.*

- *If for some $0 < t < 1$, $P(tx)/P(x) \to 1$ as $x \to \infty$, then $A(tx)/A(x) \to 1$ as $x \to \infty$.*
- *If for some $0 < t < 1$ and some $\alpha > 0$, $P(tx)/P(x) \to t^\alpha$ and $\liminf_{x\to\infty} (\log x) P(x) x^{-\alpha} > \alpha$, then $A(tx)/A(x) \to t^\alpha$ and $A(x) \sim x^\alpha A_0(x)$ with $A_0(x)$ nondecreasing and $A_0(x) \to \infty$ as $x \to \infty$. If $\limsup_{x\to\infty} (\log x) P(x) x^{-\alpha} < \alpha$, then $A(tx)/A(x) \to t^\alpha A_0(x)$ with $A_0(x)$ non-increasing and with $A_0(x) \to \infty$ as $x \to \infty$.*

The first of these results is a "multiplicative" analogue of the result appearing in item (1.4), which first appeared as a conjecture of Durrett, Granovsky and Gueron [7]. This conjecture was later proved in [2]; but the proof given there cannot be lifted to the multiplicative case despite the strong connections between partition identities and Euler products. The second result is a multiplicative analogue of the result appearing in item (1.5). Both proofs use only elementary methods.

These results are proved in Sections 4 and 5. In Sections 6 and 7, applications to logic and examples are given.

2. Background

Throughout this paper, we will use the following notation. We shall use bold uppercase letters to denote Dirichlet series.

Notation 1: Given a Dirichlet series

$$\mathbf{R}(s) = \sum_{n \geq 1} r(n) n^{-s},$$

we define the **global count function** for $\mathbf{R}(s)$ to be

$$R(x) := \sum_{n \leq x} r(n).$$

Differentiating $\mathbf{R}(s)$ gives

$$\frac{d}{ds}\mathbf{R}(s) = -\sum_{n\geq 1}(r(n)\log n)n^{-s}.$$

Notation 2: We denote by $\overline{R}(x)$ the global count function of $\frac{d}{ds}\mathbf{R}(s)$; that is,

$$\overline{R}(x) = \sum_{n\leq x} r(n)\log n.$$

We first give a simple summation lemma which will be useful to us.

LEMMA 2: *For a global count function $R(x)$ we have the formula*

$$R(x) = r(1) + \overline{R}(x)/\log x + \int_2^x \overline{R}(t)/t(\log t)^2\,dt.$$

Proof: Let $c_n = r(n)\log n$ and let $f(x) = 1/\log x$ for $x \geq 2$. Then

$$R(x) = r(1) + \sum_{2\leq n\leq x} c_n f(n).$$

Notice

$$\sum_{n\leq x} c_n = \overline{R}(x).$$

Using summation by parts (see page 346, Theorem 421 of [9]), we have

$$\begin{aligned} R(x) &= r(1) + \overline{R}(x)f(x) - \int_2^x \overline{R}(t)f'(t)dt \\ &= r(1) + \overline{R}(x)/\log x + \int_2^x \overline{R}(t)/t(\log t)^2 dt. \qquad \blacksquare \end{aligned}$$

PROPOSITION 3: *Suppose $R(x), S(x)$ are global count functions. If $\overline{R}(x) = \mathrm{o}(\overline{S}(x))$ and $S(x) \to \infty$ as $x \to \infty$, then $R(x) = \mathrm{o}(S(x))$.*

Proof: Let $\varepsilon > 0$. By assumption we can find $M > 0$ such that $\overline{R}(x) < \varepsilon\overline{S}(x)$ for all $x > M$. Using Lemma 2, we have

$$\begin{aligned} R(x) &= r(1) + \overline{R}(x)/\log x + \int_2^x \overline{R}(t)/t(\log t)^2 dt \\ &\leq r(1) + \varepsilon\overline{S}(x)/\log x + \int_2^M \overline{R}(t)/t(\log t)^2 dt + \int_M^x \varepsilon\overline{S}(t)/t(\log t)^2 dt \\ &\leq r(1) + \varepsilon\left(\overline{S}(x)/\log x + \int_2^x \overline{S}(t)/t(\log t)^2 dt\right) + \int_2^M \overline{R}(t)/t(\log t)^2 dt \\ &\leq r(1) + \varepsilon S(x) + \overline{R}(M)\int_2^\infty 1/t(\log t)^2 dt \quad \text{by Lemma 2} \\ &= \varepsilon S(x) + \mathrm{O}(1). \end{aligned}$$

Since $S(x) \to \infty$ as $x \to \infty$, we have that $R(x) = \mathrm{o}(S(x))$. ∎

We first note some basic facts about functions of **regular variation**.

Definition 1: We denote by RV_α the class of functions $F(x)$ that are eventually defined and eventually positive that satisfy

$$\lim_{x\to\infty} \frac{F(tx)}{F(x)} = t^\alpha \quad \text{for } t > 0.$$

PROPOSITION 4: *Suppose $F(x) \in \mathsf{RV}_\alpha$ and $G(x) \in \mathsf{RV}_\beta$ with $\alpha \le \beta$. Then*

- $F(x)G(x) \in \mathsf{RV}_{\alpha+\beta}$.
- $F(x) + G(x) \in \mathsf{RV}_\beta$.
- *If $\alpha > -1$ and $F(x)$ is integrable then $\int_{x_0}^x F(t)dt \in \mathsf{RV}_{\alpha+1}$.*

Proof: The first item is Proposition 7.21 (b) of [4]. For items 2 and 3 see Proposition 1.7.2 of [8]. ∎

PROPOSITION 5: *Suppose $\alpha > 0$ and $R(x)$ is a global count function. Then $R(x) \in \mathsf{RV}_\alpha$ if and only if $\overline{R}(x) \in \mathsf{RV}_\alpha$. Moreover, if $R(x) \in \mathsf{RV}_\alpha$, then $\overline{R}(x) \sim R(x)\log x$.*

First suppose that $\overline{R}(x) \in \mathsf{RV}_\alpha$. By Lemma 2 we have

$$R(x) = r(1) + \overline{R}(x)/\log x + \int_2^x \overline{R}(x)/t(\log t)^2 dt.$$

By part 1 of Lemma 4, we have $\overline{R}(x)/\log x \in \mathsf{RV}_\alpha$ and $\overline{R}(x)/x(\log x)^2 \in \mathsf{RV}_{\alpha-1}$. Using part 3 of Lemma 4, we see that $\int_2^x \overline{R}(t)/t(\log t)^2 dt \in \mathsf{RV}_\alpha$. From part 2 of Lemma 4 it follows that $R(x) \in \mathsf{RV}_\alpha$.

Now suppose that $R(x) \in \mathsf{RV}_\alpha$. We have

$$\overline{R}(x) = \sum_{n\le x} r(n)\log n \le R(x)\log x.$$

Let $0 < \gamma < 1$. Notice that $R(x^\gamma) = \mathrm{o}(R(x))$. To see this, note that $R(x)$ is nondecreasing, and thus integrable, and $R(x)$ diverges to infinity. Thus, for any $\beta < \alpha$ and $\beta' > \alpha$, we have

$$x^\beta < R(x) < x^{\beta'}$$

for x sufficiently large, by Proposition 7.23 of [4]. We choose β and β' such that $\beta < \alpha < \beta'$ and $\gamma\beta' < \beta$. Then for x sufficiently large we have $R(x^\gamma) < x^{\gamma\beta'} =$

o$(R(x))$. Thus

$$\begin{aligned}\overline{R}(x) &= \sum_{n\le x} r(n)\log n\\ &\ge \sum_{x^\gamma\le n\le x} r(n)\log(x^\gamma)\\ &= \gamma(\log x)(R(x)-R(x^\gamma))\\ &= \gamma(\log x)R(x)(1+\mathrm{o}(1)).\end{aligned}$$

Thus

$$\gamma < \liminf_{x\to\infty} \overline{R}(x)/(R(x)\log x) \le 1.$$

Since $0<\gamma<1$ is arbitrary, the result follows. ∎

3. Relations between Dirichlet series

For this section along with Sections 4 and 5, we assume we are given Dirichlet series

$$\mathbf{A}(s) = \sum_{n\ge 1} a(n)n^{-s}, \quad \mathbf{P}(s) = \sum_{n\ge 1} p(n)n^{-s}$$

satisfying the fundamental identity

$$\mathbf{A}(s) = \prod_{j\ge 1}(1-j^{-s})^{-p(j)}. \tag{3.7}$$

We find it convenient to introduce a related Dirichlet series

$$\mathbf{H}(s) = \sum_{n\ge 1} h(n)n^{-s} := \sum_{m\ge 1}\mathbf{P}(ms)/m. \tag{3.8}$$

One has $h(1)=0$ and

$$\mathbf{A}(s) = \exp(\mathbf{H}(s)), \tag{3.9}$$

$$h(n) = \sum_{a^b=n} p(a)/b \quad \text{for } n\ge 2, \tag{3.10}$$

$$H(x) = \sum_{2\le a^b\le x} p(a)/b. \tag{3.11}$$

In many of the cases we are interested in, $H(x)$ will not necessarily be well-behaved. We can often get around this problem by studying $\overline{H}(x)$ instead. We have the following fact.

LEMMA 6: $\overline{H}(x) = \sum_{2\le a\le x} p(a)\log a\lfloor \log x/\log a\rfloor$.

Proof: We have, for $n \ge 2$,

$$h(n) = \sum_{a^b=n} p(a)/b.$$

Hence

$$\begin{aligned}\overline{H}(x) &= \sum_{2\le n\le x} h(n)\log n \\ &= \sum_{2\le n\le x}\sum_{a^b=n} p(a)\log n/b \\ &= \sum_{2\le n\le x}\sum_{a^b=n} p(a)\log(a^b)/b \\ &= \sum_{2\le n\le x}\sum_{a^b=n} p(a)\log a \\ &= \sum_{2\le a\le x}\sum_{n\le \log x/\log a} p(a)\log a \\ &= \sum_{2\le a\le x} p(a)\log a\lfloor \log x/\log a\rfloor. \qquad \blacksquare\end{aligned}$$

LEMMA 7: $\overline{A}(x) = \sum_{1\le n\le x} a(n)\overline{H}(x/n)$.

Proof: By differentiating $\mathbf{A}(s) = \exp(\mathbf{H}(s))$ we have

$$\mathbf{A}'(s) = \mathbf{A}(s)\mathbf{H}'(s),$$

so

$$\sum_n (a(n)\log n)n^{-s} = \sum_n a(n)n^{-s}\cdot\sum_n (h(n)\log n)n^{-s}.$$

From this we have

$$\begin{aligned}\sum_{n\le x} a(n)\log n &= \sum_{mn\le x} a(n)h(m)\log m \\ &= \sum_{n\le x} a(n)\sum_{m\le x/n} h(m)\log m \\ &= \sum_{n\le x} a(n)\overline{H}(x/n). \qquad \blacksquare\end{aligned}$$

This fact gives the following estimates for $\overline{H}(x)$.

Proposition 8: $\frac{1}{2}P(x)\log x \le \overline{H}(x) \le P(x)\log x$.

Proof: Notice for $a \le x^{1/2}$ we have $\log x/\log a \ge 2$. Hence $\lfloor \log x/\log a \rfloor \ge \frac{1}{2}\log x/\log a$ for $a \le x^{1/2}$. Using this fact and Lemma 6, we see that

$$\begin{aligned}\overline{H}(x) &= \sum_{2\le a\le x} p(a)\log a \lfloor \log x/\log a\rfloor \\ &\ge \sum_{2\le a\le x^{1/2}} \frac{1}{2}p(a)\log a \log x/\log a + \sum_{x^{1/2}<a\le x} p(a)\log a \\ &\ge \frac{1}{2}P(x^{1/2})\log x + \big(P(x)-P(x^{1/2})\big)\log(x^{1/2}) \\ &= \frac{1}{2}P(x)\log x.\end{aligned}$$

To see the other inequality, observe that

$$\overline{H}(x) = \sum_{2\le a\le x} p(a)\log a\lfloor \log x/\log a\rfloor \le \sum_{2\le a\le x} p(a)\log x = P(x)\log x. \qquad ∎$$

In the case that we have positive abscissa of convergence and $P(x)$ is well-behaved, the behavior of $H(x)$ is asymptotically the same as $P(x)$. We make this more precise in the following proposition.

Proposition 9: *Suppose $\alpha > 0$. If $P(x) \in \mathsf{RV}_\alpha$, then $H(x) \sim P(x)$.*

Proof: We have $P(x) \le H(x)$. We also have

$$\begin{aligned}H(x) &= \sum_{2\le n\le x}\sum_{a^b=n} p(a)/b \\ &= \sum_{b\le \log x/\log 2} \frac{1}{b}\sum_{a\le x^{1/b}} p(a) \\ &= \sum_{b\le \log x/\log 2} \frac{1}{b}P(x^{1/b}) \\ &\le P(x) + \sum_{2\le b\le \log x} P(x^{1/2}) \\ &\le P(x) + (\log x)P(x^{1/2}).\end{aligned}$$

Choose β between $\alpha/2$ and α. Then for x sufficiently large we have

$$x^\beta < P(x) < x^{2\beta}$$

by Proposition 7.23 of [4]. Hence for x sufficiently large we have

$$(\log x)P(x^{1/2}) < (\log x)x^{\beta} = \mathrm{o}(P(x)).$$

Thus $H(x) \sim P(x)$. ∎

4. Proofs for $P(x) \in \mathsf{RV}_0$

In this section, we investigate the behavior of Dirichlet series with a slowly varying global count function. Our goal is to show that if $P(x) \in \mathsf{RV}_0$, then $A(x) \in \mathsf{RV}_0$. The idea is simply to differentiate the fundamental identity given in equation (3.7) and use this to analyze the behavior of $A(x)$ given that $P(x)$ satisfies certain regularity conditions.

Lemma 10: *Suppose $P(x) \in \mathsf{RV}_0$. Then for $i > 1$, we have*

$$\overline{H}(x) = \mathrm{o}\bigg(\sum_{k\geq 0} \overline{H}(x/i^k)\bigg).$$

Proof: Let $i > 1$. Since $P(x) \in \mathsf{RV}_0$, by Lemma 8 we have

$$\sum_{k\geq 0} \overline{H}(x/i^k) \geq \frac{1}{2}\bigg(\sum_{k\geq 0} P(x/i^k)\log(x/i^k)\bigg). \tag{4.12}$$

Since $P(x)\log x \in \mathsf{RV}_0$, we have

$$P(x)\log x = \mathrm{o}\bigg(\sum_{k\geq 0} P(x/i^k)\log(x/i^k)\bigg).$$

Equivalently,

$$P(x)\log x = \mathrm{o}\bigg(\sum_{k\geq 0} \overline{H}(x/i^k)\bigg).$$

Hence by Lemma 8 we have

$$\overline{H}(x) \leq P(x)\log x = \mathrm{o}\bigg(\sum_{k\geq 0} \overline{H}(x/i^k)\bigg).$$ ∎

Lemma 11: *Suppose $P(x)$ is eventually positive. Then*

$$A(x) = \mathrm{o}(\overline{A}(x)).$$

Proof: Note that for any $N > 0$ there exists an M such that $\log x > N$ for all $x \geq M$. Hence

$$\overline{A}(x) = \sum_{n \leq x} a(n) \log n \geq \sum_{M \leq n \leq x} a(n) N = (A(x) - A(M))N.$$

Thus

$$\liminf_{x \to \infty} \overline{A}(x)/A(x) \geq N,$$

since $A(x) \to \infty$ as $x \to \infty$. ∎

THEOREM 12: *If* $P(x) \in \mathsf{RV}_0$, *then* $A(x) \in \mathsf{RV}_0$.

Proof: Suppose $P(x) \in \mathsf{RV}_0$. Since the theorem is known to be true when $\sum_n p(n) < \infty$ (see Section 10.3 of [4]), we can assume $\sum_n p(n) = \infty$. Choose some m such that $p(m) \neq 0$. Define

$$\mathbf{P}_1(s) := \sum p_1(n) n^{-s} = \mathbf{P}(s) - 1/m^s$$

and

$$\mathbf{A}_1(s) := \sum_{n \geq 1} a_1(n) n^{-s} = \mathbf{A}(s)(1 - m^{-s}).$$

Clearly $P_1(x) = P(x) - 1$ for $x > m$ is in RV_0. We claim that $\overline{A_1}(x) = \mathrm{o}(\overline{A}(x))$. Let $\mathbf{H}_1(x)$ be such that $\mathbf{A}_1(x) = \exp(\mathbf{H}_1(x))$. By Lemma 7 we have

$$\overline{A_1}(x) = \sum_{n \leq x} a_1(n) \overline{H_1}(x/n) \quad \text{and} \quad \overline{A}(x) = \sum_{n \leq x} a(n) \overline{H}(x/n).$$

Clearly $\overline{H}(t) \geq \overline{H_1}(t)$ and $a(n) = \sum_{m^k | n} a_1(n/m^k)$. Hence we see

$$\begin{aligned} \overline{A}(x) &\geq \sum_{n \leq x} \overline{H_1}(x/n) \sum_{m^k | n} a_1(n/m^k) \\ &= \sum_{n \leq x} a_1(n) \Bigg(\sum_{m^k \leq x/n} \overline{H_1}(x/m^k n) \Bigg). \end{aligned}$$

Let $M > 0$. By Lemma 3 there exists some t_0 such that

$$\sum_{m^k < t} \overline{H_1}(t/m^k) > M \overline{H_1}(t)$$

for all $t > t_0$. Hence we have

$$\begin{aligned}\overline{A}(x) &\geq \sum_{n \leq x/t_0} a_1(n) M \overline{H_1}(x/n) \\ &\geq M\overline{A_1}(x) - M \sum_{x/t_0 \leq n \leq x} a_1(n)\overline{H_1}(x/n) \\ &\geq M\overline{A_1}(x) - M\overline{H_1}(t_0)A_1(x) \\ &= M\overline{A_1}(x)(1+\mathrm{o}(1)) \quad \text{by Lemma 11.}\end{aligned}$$

Hence $\overline{A_1}(x) = \mathrm{o}(\overline{A}(x))$.

Now that we have established that $\overline{A_1}(x) = \mathrm{o}(\overline{A}(x))$, the proof is straightforward. Applying Lemma 2 we see that $A_1(x) = \mathrm{o}(A(x))$. But $A_1(x) = A(x) - A(x/m)$. Hence

$$A(x/m)/A(x) \to 1.$$

Thus, by Lemma 7.25 of [4], we see that $A(x) \in \mathsf{RV}_0$. ∎

5. Proofs for $P(x) \in \mathsf{RV}_\alpha$ with $\alpha > 0$

In this section we wish to prove an analogue of the main result proved in the preceding section; namely that if $P(x) \in \mathsf{RV}_0$ then $A(x) \in \mathsf{RV}_0$. Unfortunately, there is greater difficulty in this case, which arises from the fact that if $\alpha > 0$ and $P(x) \in \mathsf{RV}_\alpha$, then $\mathbf{A}(s)$ may converge at $s = \alpha$. This possibility cannot occur when $\alpha = 0$ since we insist that the coefficients of our Dirichlet series are nonnegative integers. With an additional hypothesis, however, the difficulty can be handled, and we are able to prove the second item appearing in the statement of Theorem 1.

Lemma 13: *Suppose $\alpha > 0$. If $P(x) \in \mathsf{RV}_\alpha$, then $H(x) \sim P(x)$ and $A(x) \in \mathsf{RV}_\alpha$.*

Proof: We have $\overline{H}(x) \in \mathsf{RV}_\alpha$ by Proposition 5. Let $\varepsilon > 0$. We can find $M > 1$ such that

$$|\overline{H}(2x) - 2^\alpha \overline{H}(x)| < \varepsilon \overline{H}(x)$$

for all $x > M$. Then we have

$$\begin{aligned}|\overline{A}(2x) - 2^\alpha \overline{A}(x)| &= \left| \sum_{n \leq 2x} (\overline{H}(2x/n) - 2^\alpha \overline{H}(x/n)) a(n) \right| \quad \text{by Lemma 7} \\ &\leq \sum_{n \leq 2x/M} \varepsilon \overline{H}(x/n) a(n)\end{aligned}$$

$$+\sum_{2x/M<n\le 2x}|\overline{H}(2x/n)-2^{\alpha}\overline{H}(x/n)|a(n)$$
$$\le\varepsilon\sum_{n\le 2x}\overline{H}(x/n)a(n)+\overline{H}(M)(1+2^{\alpha})\sum_{n\le 2x}a(n)$$
$$=\varepsilon\overline{A}(2x)+\overline{H}(M)(1+2^{\alpha})A(2x)\quad\text{by Lemma 7}$$
$$=\varepsilon\overline{A}(2x)+\mathrm{o}(\overline{A}(2x))\quad\text{by Lemma 11.}$$

Hence $\overline{A}(x)\in\mathsf{RV}_{\alpha}$. By Proposition 5 we have that $A(x)\in\mathsf{RV}_{\alpha}$. ∎

The next lemma says that any slowly varying function is asymptotic to a step function, where the steps occur on intervals of the form $[\beta^n,\beta^{n+1})$.

Lemma 14: *Suppose $F(x)\in\mathsf{RV}_0$ and $\beta>1$. Then*

$$F(x)\sim F(\beta^{\lfloor\log_\beta x\rfloor}).$$

Proof: Let $\beta_x=x/\beta^{\lfloor\log_\beta x\rfloor}$. Then clearly $\beta_x\in[1,\beta]$ for $x>0$, and then

$$F(x)=F(\beta_x\cdot\beta^{\lfloor\log_\beta x\rfloor})\sim F(\beta^{\lfloor\log_\beta x\rfloor})$$

by the Uniform Convergence Theorem for functions of regular variation (Theorem 1.3 of [8]). ∎

Theorem 15: *Suppose $\alpha>0$ and suppose $P(x)\sim x^{\alpha}P_0(x)/\log x$ for some nondecreasing function $P_0(x)\in\mathsf{RV}_0$ satisfying*

$$\lim_{x\to\infty}P_0(x)\in(\alpha^{-1},\infty].$$

Then $A(x)\sim x^{\alpha}A_0(x)$ where $A_0(x)$ is a nondecreasing function in RV_0 which tends to infinity as x tends to infinity.

Proof: Notice that since $P_0(x)$ tends to a value greater than $1/\alpha$, we can alter the initial values of $P_0(x)$ if necessary so that $P_0(x)$ is still nondecreasing and satisfies:

$$P_0(t)=0\quad\text{for } t<1$$

and

$$P_0(1)=L>1/\alpha.$$

By Lemma 7 we have

$$\overline{A}(x)=\sum_{n\le x}\overline{H}(x/n)a(n).$$

With this in mind we define

$$\Psi(x) := \sum_{n\le x} P_0(x/n)a(n)n^{-\alpha}. \tag{5.13}$$

Claim 1: $\Psi(x) \sim \overline{A}(x)x^{-\alpha}$.

Let $\varepsilon > 0$. By Lemma 13 and Proposition 5 we have

$$\overline{H}(x) \sim P(x)\log x \sim x^{\alpha}P_0(x).$$

Hence we can find M such that

$$|t^{\alpha}P_0(t) - \overline{H}(t)| < \varepsilon\overline{H}(t)$$

for $t > M$. Choose C such that

$$|t^{\alpha}P_0(t) - \overline{H}(t)| < C$$

for $t \le M$. Then, by Lemma 6,

$$\begin{aligned}|x^{\alpha}\Psi(x) - \overline{A}(x)| &= \sum_{n\le x}((x/n)^{\alpha}P_0(x/n) - \overline{H}(x/n))a(n)\\ &\le \sum_{n\le x/M}\varepsilon\overline{H}(x/n)a(n) + \sum_{x/M\le n\le x} Ca(n)\\ &\le \varepsilon\overline{A}(x) + CA(x).\end{aligned}$$

By Lemma 13, we have that $A(x) \in \mathsf{RV}_{\alpha}$, as $P(x) \in \mathsf{RV}_{\alpha}$. Thus $\overline{A}(x) \sim (\log x)A(x)$ by Proposition 5 and in particular, $A(x) = \mathrm{o}(\overline{A}(x))$. It follows that $\Psi(x) \sim \overline{A}(x)x^{-\alpha}$. This proves the claim. ∎

We now have

$$\Psi(x) \sim \overline{A}(x)x^{-\alpha} \sim A(x)x^{-\alpha}\log x. \tag{5.14}$$

Also $H(x) \sim P(x)$ by Lemma 13. Thus $H(x) \sim x^{\alpha}P_0(x)/\log x$ and so

$$\overline{H}(x) \sim H(x)\log x \sim x^{\alpha}P_0(x).$$

Notice for $t > 1$,

$$\begin{aligned}\Psi(tx) - \Psi(x) &= \sum_{n\le x}(P_0(tx/n) - P_0(x/n))a(n)n^{-\alpha} + \sum_{x<n\le tx} P_0(tx/n)a(n)n^{-\alpha}\\ &\ge \sum_{x<n\le tx} P_0(1)a(n)n^{-\alpha}\\ &\ge L(A(tx) - A(x))(tx)^{-\alpha}\\ &= LA(x)x^{-\alpha}(t^{\alpha}-1)t^{-\alpha} + \mathrm{o}(A(x)x^{-\alpha}).\end{aligned}$$

Notice

$$\lim_{t\to 1^+} L(t^\alpha - 1)t^{-\alpha}/\log t = L\alpha > 1.$$

Hence we can find $\beta > 1$ and $C > 1$ such that

$$\Psi(\beta x) - \Psi(x) \geq C(\log\beta)A(x)x^{-\alpha}$$

for all x sufficiently large. Now by (5.14)

$$A(x)x^{-\alpha} \sim \Psi(x)/\log x.$$

Hence if we choose $1 < C' < C$, we have that

$$\Psi(\beta x) - \Psi(x) \geq C'(\log\beta)\Psi(x)/\log x \tag{5.15}$$

for all x sufficiently large. We define

$$\Phi(x) := \Psi(x)/\log x \sim A(x)x^{-\alpha}. \tag{5.16}$$

Then we can rewrite (5.15) as

$$\Phi(\beta x)\log(\beta x) - \Phi(x)\log x \geq C'(\log\beta)\Phi(x).$$

Or equivalently,

$$(\Phi(\beta x) - \Phi(x))\log x \geq C'(\log\beta)\Phi(x) - (\log\beta)\Phi(\beta x).$$

Now $(\log\beta)\Phi(\beta x) \sim (\log\beta)\Phi(x)$. Since $C' > 1$, we can conclude that $\Phi(x) \sim A(x)x^{-\alpha}$ has the property that

$$\Phi(\beta x) - \Phi(x) > C''\Phi(x)/\log x > 0$$

for some positive constant C'', for all $x > N_0$ for some N_0. By Lemma 14 we see that $\Phi(x)$ is asymptotic to a nondecreasing function $A_0(x)$. Clearly $A(x) \sim x^\alpha A_0(x)$, and as $\Phi(x)$ is in RV_0, so is $A_0(x)$. Notice also that since $\Phi(\beta x) > \Phi(x)$ for $x > N_0$, we have

$$\Phi(\beta^d x) - \Phi(x) > C''d\Phi(x)\left(\sum_{j=1}^{d}(\log(\beta^j x))^{-1}\right) \tag{5.17}$$

for all $x > N_0$ and all integers $d \geq 1$. Further,

$$\begin{aligned}\sum_{j=1}^{d}(\log(\beta^j x))^{-1} &= \sum_{j=1}^{d}(j\log\beta + \log x)^{-1}\\ &\leq \int_0^d (t\log\beta + \log x)^{-1}\,dt\\ &= \log(d\log\beta + \log x)/\log(\beta) - \log\log x/\log(\beta)\\ &\leq \log(d\log\beta)/\log(\beta).\end{aligned}$$

Using this fact along with equation (5.17), we see that

$$\Phi(\beta^d x) - \Phi(x) > C'' d \log \beta \Phi(x)/(\log(d \log(\beta)).$$

Now if we fix $x > N_0$ and let $d \to \infty$, then we see that $\Phi(\beta^d x) \to \infty$ as $d \to \infty$. Hence $A_0(x)$ must tend to infinity as x tends to infinity, since it is nondecreasing. ∎

Remark 16: By examining the proof of Proposition 1.3.4 of [3] (due to Adamović) it is clear that one can also choose the $A_0(x)$ of Theorem 15 to be a C^∞ function.

We now give an example which shows that the condition given in the statement of Theorem 15 cannot be relaxed.

Example 1: The conclusion in the statement of Theorem 15 is not necessarily true if we assume instead that

$$\lim_{x\to\infty} P_0(x) \in (\beta, \infty]$$

for some $\beta < \alpha$.

Suppose $\alpha > 0$ and let $N > 2$ be an integer. Choose an integer m such that $m\alpha > 1$. Take $p(n) = \lfloor p^{\alpha - 1/m} \rfloor$ if n is of the form p^m for some prime p not congruent to 1 mod N, and take $p(n) = 0$ otherwise.

Claim 2: $P(x) \sim (1 - 1/\phi(N))x^\alpha/(\alpha \log x)$.

Proof: We use summation by parts, taking $c_n = 1$ if $n = p^m$ for some prime p which is not congruent to 1 mod N. We take $f(x) = x^{\alpha - 1/m}$. Notice that

$$P(x^m) = \sum_{j \le x^m} p(j) = \sum_{j \le x^m} c_j \lfloor f(j) \rfloor = \sum_{j \le x^m} c_j f(j) + \mathrm{O}(x). \tag{5.18}$$

Let $C(t) = \sum_{j \le t} c_j$. Using summation by parts, we see that

$$\sum_{j \le x^m} c_j f(j) = C(x^m) x^{m\alpha - 1} - (\alpha - 1/m) \int_1^{x^m} C(t) t^{\alpha - 1 - 1/m} dt. \tag{5.19}$$

Let $\epsilon > 0$. By Dirichlet's theorem, there is some $M > 0$ such that

$$|C(t) - m(1 - 1/\phi(N)) t^{1/m}/\log(t)| < \epsilon t^{1/m}/\log(t)$$

for all $t > M$. Thus

$$C(x^m) x^{m\alpha - 1} = (1 - 1/\phi(N)) x^{m\alpha}/\log(x) + \mathrm{o}(x^{m\alpha}/\log(x)).$$

Also,

$$\int_1^{x^m} C(t)t^{\alpha-1-1/m} = (1+\mathrm{o}(1))\int_1^{x^m} t^{\alpha-1/m}/\log(t)dt + \mathrm{O}(1).$$

An easy argument shows that

$$\int_1^u t^\beta/\log(t)dt \sim u^{\beta+1}/((\beta+1)\log(u)),$$

as $u \to \infty$. Thus

$$C(x^m)x^{m\alpha-1} - (\alpha-1/m)\int_1^{x^m} C(t)t^{\alpha-1-1/m}dt = (1-1/\phi(N))x^{m\alpha}/m\alpha\log(x).$$

The result follows using equations (5.18) and (5.19). ∎

Let $P_0(x) = 1 - 1/\phi(N)$. By letting N grow large, note that we can get $(1-1/\phi(N))$ as close to 1 as we please.

Claim 3: *$A(x)/x^2 \to 0$ as $x \to \infty$.*

Proof: Observe that

$$\begin{aligned}[p^{-kms}](1-p^{\alpha-1/m}/p^{ms})^{-1} &= p^{(\alpha-1/m)k}\\ &\geq \binom{-\lfloor p^{\alpha-1/m}\rfloor}{k}(-1)^k\\ &= [p^{-kms}](1-p^{-ms})^{-\lfloor p^{\alpha-1/m}\rfloor}.\end{aligned}$$

Hence

$$\begin{aligned}a(n^m) &= [n^{-ms}]\prod_{p\not\equiv 1 \bmod N}(1-p^{-ms})^{-\lfloor p^{\alpha-1/m}}\\ &\leq [n^{-ms}]\prod_{p\not\equiv 1 \bmod N}(1-p^{\alpha-1/m}/p^{ms})^{-1}.\end{aligned}$$

Thus we see that $a(n^m) = 0$ if n has a prime factor that is congruent to $1 \bmod N$; moreover, if n has no prime factors congruent to 1 mod N, then

$$a(n^m) \leq [n^{-ms}]\prod_p(1-p^{\alpha-1/m}/p^{ms})^{-1} = [n^{-ms}]\zeta(ms-\alpha+1/m) = n^{\alpha-1/m}.$$

Now

$$A(x^m) = \sum_{n\leq x} a(n^m)$$

and so $A(x^m)$ is bounded above by

$$x^{m\alpha-1}\cdot \mathrm{Card}\{n \leq x\colon n \text{ has no prime factors congruent to } 1 \bmod N\}.$$

It is a well-known consequence of Dirichlet's theorem on primes in an arithmetic progression that the set of numbers less than or equal to x having no prime factors congruent to 1 mod N is $o(x)$ as $x \to \infty$. The result follows. ∎

Hence $A(x)$ cannot be asymptotic to $x^2 A_0(x)$ with $A_0(x)$ nondecreasing. This shows that the conclusion appearing in the statement of Theorem 15 will not necessarily hold if we assume that $\lim_{x\to\infty} P_0(x) < \alpha$.

6. Applications to logical limit laws

Now we look at how this result can be used to find first-order 0–1 laws. From Chapter 12 of [4], a class $\mathcal{K}$ of finite first-order structures is **adequate** if the class $\mathcal{K}$ is closed under formation of direct products and the extraction of indecomposable factors, and, up to isomorphism, every member of $\mathcal{K}$ can be uniquely factored into irreducible members of $\mathcal{K}$ and there is some element $E \in \mathcal{K}$ such that $E \times X \cong X$ for all $X \in \mathcal{K}$ (such an E is unique up to isomorphism). Finite Abelian groups provide a popular example of an adequate class. In this case, we see that the irreducible members of this class are the cyclic p-groups, with p some prime. The fundamental theorem of finitely generated abelian groups shows that every finite abelian group can be uniquely written (up to a permutation) as a product of cyclic p-groups. $\mathcal{K}$ has a **first-order 0–1 law** if for each first-order sentence φ the class $\mathcal{K}_\varphi$ of members of $\mathcal{K}$ satisfying φ has an asymptotic density in $\mathcal{K}$ that is either 0 or 1; that is, if φ is a first-order sentence, then the proportion of members of $\mathcal{K}$ of size less than or equal to n which satisfy φ must go to either 0 or 1 as $n \to \infty$. For example, the class of finite abelian groups (up to isomorphism) does not have a first-order 0–1 law, because the sentence $\{g^2 = 1 \text{ for some } g \neq 1\}$ is satisfied by a group G if and only if G has even order. The proportion of groups of size less than or equal to n of even order is asymptotic to the middle term of the expression

$$0 < 1 - \prod_{j\geq 1}(1 - 1/2^j) < 1,$$

as $n \to \infty$ (see Section 8.3.5 on page 148 of [4]). More generally, we say that $\mathcal{K}$ has a **first-order limit law** if for any first-order sentence φ, the limit as n tends to infinity of the proportion of members of $\mathcal{K}$ of size less than or equal to n which satisfy φ exists. The limiting value will be some value between 0 and 1.

Given an adequate class $\mathcal{K}$, we let $\mathsf{P}_{\mathcal{K}}(x)$ denote the number of irreducible members of $\mathcal{K}$ of size less than or equal to x and we let $\mathsf{A}_{\mathcal{K}}(x)$ denote the number of members of $\mathcal{K}$ of size less than or equal to x. Using the results we have developed so far, we have the following results.

THEOREM 17: *Let $\mathcal{K}$ be an adequate class. If $\mathsf{P}_{\mathcal{K}}(x) \in \mathsf{RV}_0$ then $\mathcal{K}$ has a first-order 0–1 law.*

Proof: From Theorem 12 we know that $\mathsf{A}_{\mathcal{K}}(x) \in \mathsf{RV}_0$. Thus Proposition 12.19 (a) of [4] applies. ∎

THEOREM 18: *Let $\mathcal{K}$ be an adequate class. If $\mathsf{P}_{\mathcal{K}}(x)$ satisfies the hypotheses of Theorem 15, then $\mathcal{K}$ has a first-order limit law.*

Proof: From Theorem 15 we know that $\mathsf{A}_{\mathcal{K}}(x) \sim x^{\alpha}\mathsf{A}_0(x)$ with $\mathsf{A}_0(x)$ a nondecreasing function in RV_0. Thus Corollary 11.17 and Theorem 12.8 of [4] apply. ∎

Theorem 17 is Conjecture 10.7 of [4]. Theorem 18 is very similar to Conjecture 11.26 of [4], which has the hypothesis

$$\liminf_{x\to\infty} P(x)\log x/x^{\alpha} > 1.$$

In light of Theorem 15, however, it seems that the more natural hypothesis is

$$\liminf_{x\to\infty} P(x)\log x/x^{\alpha} > 1/\alpha.$$

These results are useful, because, previously, to deduce limit laws based on estimates for $P(x)$, one needed to use difficult asymptotic methods to find the asymptotic behavior of $A(x)$. Our approach is more elementary and avoids finding the asymptotic behavior of $A(x)$, instead just verifying necessary conditions.

7. Examples and comparisons with previous results

Theorems 17 and 18 state that under certain conditions on an adequate class of objects, one can conclude that the class has a first-order logical limit law. We point out that our conditions are entirely concerned with the asymptotic behavior of $P(x)$, the global count function for the irreducible elements of our class. Knopfmacher [10], in his study of Prime Number Theorems for number systems, gave conditions which are concerned with the behavior of $A(x)$, the global count function of all elements of the class, when $\alpha > 0$; and with $P(x)$ when $\alpha = 0$. Although Knopfmacher's conditions were designed for Prime Number Theorems, they proved to be a valuable source for examples in the development of the theory of asymptotic density for logical limit laws in the 1990s.

The conditions given in Burris' book [4] for the study of asymptotic density in multiplicative number systems are all concerned with the behavior of $A(x)$, with one exception which puts very strong conditions on the local count function $p(n)$ of the irreducible members of the class. We will explain in detail that, for the purpose of asymptotic density, some of the conditions previously used (such as Knopfmacher's Axioms A and C) have now been superseded by more general conditions of the author; and the other conditions previously used have not been able to provide examples not covered by the authors conditions. Furthermore, each of the author's conditions provides examples that cannot be derived from the conditions used previously.

7.1 Discussion of Theorem 17. Let $\mathcal{K}$ be an adequate class of finite objects and let α denote the abscissa of convergence of the corresponding Dirichlet series. In chapter 10, page 198 of [4], the basic result is given that if $\alpha = 0$, then $A(x) \in \mathsf{RV}_0$ is a necessary and sufficient condition for all partition sets to have asymptotic density; this, in turn, gives 0–1 laws. Consequently, the goal is to find intermediate results to test for "$A(x) \in \mathsf{RV}_0$". Burris [4] uses the polylog condition from Bell [1] that says if $P(x) = \mathrm{O}((\log x)^d)$, for some $d \geq 0$, then $A(x) \in \mathsf{RV}_0$; moreover, every other condition he gives to ensure that $A(x) \in \mathsf{RV}_0$ follows from this one. We note that Theorem 17 does not follow from this result. Indeed, any $P(x)$ which satisfies

$$P(x) \sim C(\log\log x)^{\alpha}(\log x)^{\beta} e^{(\log x)^{\gamma}}, \quad C > 0, \quad \alpha, \beta \geq 0, \quad 0 < \gamma < 1,$$

satisfies $P(x) \in \mathsf{RV}_0$, but for any d does not satisfy $P(x) = \mathrm{O}((\log x)^d)$.

Throughout this paper we have used the notation of Burris [4]. The notation used by Knopfmacher [10] is quite different, and so we translate his results into the notation we are using. Knopfmacher's $\pi^{\#}(x)$ corresponds to our $P(b^x)$, where b is some number which depends on the semigroup being studied; for instance, for Abelian p-groups, $b = p$. Axiom C on page 221 of Knopfmacher [10] is the same as stating

$$P(b^x) \sim C x^{\kappa} (\log x)^{\nu}, \quad C, \kappa > 0, \ \nu \,\mathrm{real}\,.$$

Equivalently,

$$P(x) \sim C(\log x)^{\kappa}(\log\log x)^{\nu}, \quad C, \kappa > 0, \ \nu \,\mathrm{real}\,.$$

He applies this result to (the finite members of) Abelian p-groups, semisimple p-rings, and other algebraic structures to deduce Prime Number Theorems.* Knopfmacher shows that if Axiom C holds,

$$N^{\#}(x) = \exp((c + \mathrm{o}(1))x^{\kappa/(\kappa+1)}(\log x)^{\nu/(\kappa+1)}),$$

which is equivalent, in our notation, to

$$A(x) = \exp((c + \mathrm{o}(1))(\log x)^{\kappa/(\kappa+1)}(\log\log x)^{\nu/(\kappa+1)}).$$

This $A(x)$ is a nice RV_0 function, and thus it gives us 0–1 laws. However, note that this use of Axiom C is a very special case of Theorem 17, since $P(x) \in \mathsf{RV}_0$ if Axiom C is satisfied. Of course, Knopfmacher's result has the nice additional feature of giving the asymptotic behavior of $A(x)$, but for the purposes of asserting that a 0–1 law holds, we only need to know that $A(x) \in \mathsf{RV}_0$. We now give an example which uses Theorem 17 and which cannot be obtained from either the use of Axiom C of Knopfmacher or the polylog results used by Burris [4].

Example 2: An application to discriminator varieties. There has been considerable interest in the universal algebra community in discriminator varieties of the form $\mathcal{V} = V(\mathcal{K}^t)$ where $\mathcal{K}$ is a well known class of algebras closed under isomorphism, subalgebras and ultraproducts, in particular for $\mathcal{K}$ being (the finite members of) a well known variety such as the variety of Boolean algebras, Abelian groups, or unary algebras; and where t defines the ternary discriminator function on each member of $\mathcal{K}$ (see [5], [6], [11], [12]). Such $\mathcal{V}$ have been particularly investigated with regard to first-order decidability, and they are excellent candidates for the study of limit laws by the methods of this paper since their finite members form an adequate class of algebras, and one has the count function $P_{\mathcal{V}}(x)$ for the indecomposables of $\mathcal{V}$ being the same as the count function $A_{\mathcal{K}}(x)$ of $\mathcal{K}$. (Sometimes $\mathcal{V}$ is referred to as the discriminator variety with stalks from $\mathcal{K}$ because of the sheaf representation for members of $\mathcal{V}$.) As a result of this fact, we see that if $A_{\mathcal{K}}(x) \in \mathsf{RV}_0$, then $A_{\mathcal{V}}(x) \in \mathsf{RV}_0$. In particular, we deduce the following result.

THEOREM 19: *If $A_{\mathcal{K}}(x) \in \mathsf{RV}_0$, then the class $\mathcal{V}$ has a first-order 0–1 law.*

As an application of this fact, note that given a prime q, by Theorem 17 the finite members of the discriminator variety with stalks from abelian q-groups

* We note that Knopfmacher calls a theorem that deduces the asymptotics of $P(x)$ from the behavior of $A(x)$, *or vice versa*, a Prime Number Theorem.

has a first-order 0–1 law. To see this, notice that the number of indecomposable Abelian q-groups of size q^n is 1 (up to isomorphism), and so if $\mathcal{K}$ is the class of finite Abelian q-groups (up to isomorphism), then $P_{\mathcal{K}}(x) = \lfloor \log_q x \rfloor \in \mathsf{RV}_0$. Thus the class $\mathcal{K}$ has a growth function in RV_0 and we deduce the result from the comments above. This result cannot be obtained using previous methods.

7.2 Discussion of Theorem 18. Sárközy's Theorem ([4], page 296) is *the* fundamental tool that we rely on to prove limit laws that are not 0–1 laws; but it has been a big challenge to figure out when a number system satisfies its conditions. The main theme has been to find intermediate conditions that are possible to work with, and which imply Sárközy's conditions. There are three such results given by Burris [4]; namely, Corollary 11.17, Corollary 11.9, and a generalization of the asymptotic results of Oppenheim on the number of factorizations of an integer. All the examples in [4] are based on these three results, and not on directly using Sárközy's theorem. The key property used by Knopfmacher in his study of prime number theorems when $\alpha > 0$ is his Axiom A ([4], page 298). This condition is covered by the hypothesis of Corollary 11.19 of [4].

Axiom A of Knopfmacher, Corollary 11.17 and Corollary 11.19 of [4] all use hypotheses on $A(x)$ to verify that the conditions of Sárközy hold. The generalization of Oppenheim's result gives a beautiful asymptotic formula using the following hypothesis on the local count function $p(n)$:

$$p(n) = an^{\mu} + \mathrm{O}(n^{\eta}), \quad a > 0,\ \mu > -1,\ \text{and}\ \eta < \mu. \tag{7.20}$$

The approach given in this paper is entirely through conditions on the global count function $P(x)$. The hypotheses of Theorem 18 are easily seen to cover the above condition (7.20) used in [4]. Now we give an example to show Theorem 18 is more general.

Example 3: The (finite members of the) variety of rectangular bands has a first-order limit law, as does the (finite members of the) discriminator variety with stalks from rectangular bands.

Proof: A **rectangular band** is an idempotent semigroup satisfying the identity $xyx = x$. Every rectangular band can be uniquely decomposed into a product of a semigroup satisfying the identity $xy = x$ and a semigroup satisfying the identity $xy = y$ (such semigroups are called left zero bands and right zero bands, respectively). Given a finite left zero band, it factors uniquely into a product of left zero bands of prime size. Similarly for right zero bands. Hence if $P(x)$ is the number of irreducible rectangular bands of size at most x, then $P(x) \sim 2\pi(x) \sim 2x/\log, x$, by the prime number theorem. We see that the conditions of Theorem 18 are satisfied and so the class of finite rectangular bands has a first-order limit law. We note that the asymptotic results generalizing Oppenheim do not apply in this case. One could, however, with a little work find the asymptotic behavior of the function $A(x)$, the total number of rectangular bands of size at most x, and deduce that a limit law exists. Our approach avoids this effort, relying instead on the prime number theorem. This application to rectangular bands is therefore not new, but we present it as an example of how to apply Theorem 18. We now give a genuinely new result. Notice that by Theorem 18, we have that $A(x) \in \mathsf{RV}_1$; moreover, $A(x) \geq P(x)$ and so

$$\liminf_{x\to\infty}(\log x)A(x)/x \geq \liminf_{x\to\infty}(\log x)P(x)/x \geq 2.$$

Thus if one forms the discriminator variety $\mathcal{V}$ with stalks from finite rectangular bands, then one sees using the fact that $P_{\mathcal{V}}(x)$ is the same as $A(x)$ (as discussed in the lines before Theorem 19) that this class, too, has a first-order limit law. This result cannot be obtained using previous methods.

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